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METHOD OF REDUCTION TO THE ORDINARY DIFFERENTIAL EQUATIONS OF L. V. KANTOROVICH AND A GENERAL METHOD FOR THE SOLUTION OF MULTIDIMENSIONAL HEAT-TRANSFER EQUATIONS

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UDC 536.2:517.946

A method is proposed for the solution of multidimensional heat-transfer problems, representing a further elaboration and generalization of projection methods.

The mathematical investigation of heat- and mass-transfer processes in various kinds of heat-exchange equipment is known to require the solution of complex multidimensional problems. The advent of the Ritz and Bubnov–Galerkin methods for the solution of problems in the variational and differential formulations, respectively, set the stage for the development of a powerful trend in applied mathematics, viz., projection methods [1, 2], and afforded the conceptual possibility of solving a broad category of multidimensional problems. However, even in cases where the theory guarantees convergence of the indicated methods, a sufficiently accurate solution is obtainable, as a rule, for a large number n of pertinent parameters. This fact, in turn, means the application of computing hardware. Familiar difficulties may also be encountered in connection with the onset of instability and, accordingly, a loss of accuracy of the solution with increasing value of n (contrary to theory), up to the point of complete divergence of the process [1]. Coping with these difficulties by refinement of the coordinate functions through their orthogonalization, compliance with boundary conditions, etc., is not too effective in the general case, because all of these procedures are implemented on a distinctly a priori choice of basis. A cardinal solution of the general problem can be achieved by seeking to obtain reliable (not a priori!) functional information with essential reliance on the original mathematical statement of the problem. A first step in this direction is offered by the method of reduction to Kantorovich–Vlasov ordinary differential equations [3, 4]. In this method the constant coefficients involved in the Ritz (Galerkin) procedure are superseded by functional coefficients depending on one of the arguments of the problem, i.e., the required solution $u(\bar{x})$ of an N-dimensional problem is represented in the form

$$u(\bar{x}) = F(\bar{K}(x_k),\ \bar{\varphi}(\bar{x})). \tag{1}$$

Here $\bar{\varphi}(\bar{x}) = \{\varphi_m(\bar{x})\}_{m=1}^{l}$ is a vector function of a vector argument $\bar{x} = (x_1, x_2, \ldots, x_N)$, the components of which are basic functions selected a priori; $\bar{K}(x_k) = \{K_j(x_k)\}_{j=1}^{n}$, vector function of a variable x_k, the components of which are evaluated deterministically from the one-dimensional problem; and F, function characterizing the form of representation of the solution, i.e., its structure. It is customarily assumed in projection methods that

$$u(\bar{x}) = \sum_{m=1}^{n} K_m(x_k)\,\varphi_m(\bar{x}). \tag{1a}$$

This approach improved the convergence of the solution in comparsion with the Ritz and Bubnov–Galerkin procedures. However, because of the intuitive choice of functional informa-

Kiev Polytechnic Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 42, No. 6, pp. 1007-1013, June, 1982. Original article submitted March 31, 1980.

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tion from the remaining (N − 1) variables contained in the basis functions $\{\varphi_m(\bar{x})\}_{m=1}^{l}$, the convergence of the Kantorovich–Vlasov method is still unsatisfactory, especially for N > 2. The next step in the logical development of this approach seems natural: to introduce into the form of representation (1), (1a) as functional coefficients, not one $\bar{K}(x_k)$, but N vector functions in each of the arguments of the problem:

$$u(\bar{x}) = F(\bar{X}_1(x_1),\ \bar{X}_2(x_2),\ \ldots,\ \bar{X}_N(x_N),\ \bar{\varphi}(\bar{x})) \tag{2}$$

specifically

$$u(\bar{x}) = \sum_{m=1}^{n} X_{1m}(x_1)\, X_{2m}(x_2)\ \ldots\ X_{Nm}(x_N)\, \varphi_m(\bar{x}). \tag{2a}$$

We emphasize the fact that in this representation each vector function $\bar{X}_i(x_i) = \{X_{ij}(x_i)\}_{j=1}^{n}$ (i = 1, 2, ..., N) depends only on one variable x_i. It is clear that for the deterministic evaluation of all $X_{ij}(x_i)$ (j = 1, 2, ..., n, i = 1, 2, ..., N) it is now required at once to formulate not one, as in the Kantorovich–Vlasov method, but N one-dimensional problems. To facilitate the comparative characterization of the methods we refer to the correspondence between the number of dimensions of the original problem (N) and the number of one-dimensional problems to be solved (m) as the structure of the correspondence of N to m (N → m). By this definition the Ritz (Galerkin) and the Kantorovich–Vlasov methods have N → 0 and N → 1 structures, respectively. In this article, by contrast, we propose a method based on the representation of the solution in the form (2), realizing an N → N correspondence structure.

We outline the essentials of the method in the example of an N-dimensional problem formulated in the differential setting with respect to a scalar function $u(\bar{x})$ of a vector argument $\bar{x} = (x_1, x_2, \ldots, x_N)$:

$$Du(\bar{x}) = f(\bar{x}) \quad \bar{x} \in g, \tag{3}$$

$$Eu(\bar{x}) = \psi(\bar{x}) \quad \bar{x} \in s. \tag{4}$$

Here D and E are linear or nonlinear differential operators; g, domain with boundary s in the space R_N of the variable $\bar{x}$; and f, ψ, given functions.

As mentioned, the required solution $u(\bar{x})$ is assumed to have the form (2). In this expression the functions $\{\varphi_m(\bar{x})\}_{m=1}^{l}$ play the role of basic functions and, as in all projection methods, are selected a priori; the functions $\{X_{ij}(x_i)\}_{j=1}^{n_i}$ (i = 1, 2, ..., N), which are argument functions, are unknown and must be determined. Inasmuch as the unknowns in the proposed method comprise not just one, but a set of N vector functions, for their determinations, as we indicated, it is totally natural to formulate N one-dimensional problems in each variable of the domain. To find these problems we can invoke the fundamental notions of projection methods embodied in such classical approaches as the method of moments, the Bubnov–Galerkin method, the method of orthogonal projections, collocation, etc. [2, 5, 6]. Thus, according to the tenets of the method of orthogonal projections, a one-dimensional problem in the variable x_k is obtained from the original problem (3), (4) by substituting (2) therein, multiplying by a certain system of n_k functions $W_{kp}(\bar{x})$ (p = 1, 2, ..., n_k), and integrating with respect to all variables except x_k between their limits of variation. Applying this reasoning to each of the N arguments separately, we obtain the system of one-dimensional problems

$$\int_{x_{20}}^{x_{21}}\int_{x_{30}}^{x_{31}}\cdots\int_{x_{N0}}^{x_{N1}} [DF(\bar{X}_1(x_1),\ \bar{X}_2(x_2),\ \ldots,\ \bar{X}_N(x_N),\ \bar{\varphi}(\bar{x})) - f(\bar{x})]\, W_{1h}\, dx_2 dx_3\ \ldots\ dx_N = 0, \quad x_{10} < x_1 < x_{11}, \tag{5.1}$$

$$\int_{x_{20}}^{x_{21}}\int_{x_{30}}^{x_{31}}\cdots\int_{x_{N0}}^{x_{N1}} [EF(\bar{X}_1(x_1),\ \bar{X}_2(x_2),\ \ldots,\ \bar{X}_N(x_N),\ \bar{\varphi}(\bar{x})) - \psi(\bar{x})]\, W_{1h}\, dx_2 dx_3\ \ldots\ dx_N\Big|_{\substack{x_1=x_{10}\\ x_1=x_{11}}} = 0, \quad (k = 1, 2, \ldots, n_1),$$

$$\int_{x_{10}}^{x_{11}}\int_{x_{30}}^{x_{31}}\cdots\int_{x_{N0}}^{x_{N1}} [DF(\bar{X}_1(x_1),\ \bar{X}_2(x_2),\ \ldots,\ \bar{X}_N(\varphi_N),\ \bar{\varphi}(\bar{x})) - f(\bar{x})]\, W_{2h}\, dx_1 dx_3\ \ldots\ dx_N = 0, \quad x_{20} < x_2 < x_{21}, \tag{5.2}$$

$$\int_{x_{10}}^{x_{11}}\int_{x_{30}}^{x_{31}}\cdots\int_{x_{N0}}^{x_{N1}} [EF(\bar{X}_1(x_1),\ \bar{X}_2(x_2),\ \ldots,\ \bar{X}_N(x_N),\ \bar{\varphi}(\bar{x})) - \psi(\bar{x})]\, W_{2h}\, dx_1 dx_3 \ldots dx_N\Big|_{\substack{x_2=x_{20}\\ x_2=x_{21}}} = 0, \quad (k = 1, 2, \ldots, n_2),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{5}$$

$$\int_{x_{10}}^{x_{11}}\int_{x_{20}}^{x_{21}}\cdots\int_{x_{N-1,0}}^{x_{N-1,1}}[DF(\overline{X}_1(x_1),\ \overline{X}_2(x_2),\ \ldots,\ \overline{X}_N(x_N),\ \overline{\varphi}(\overline{x}))-$$

$$-f(\overline{x})]\,W_{Nk}\,dx_1dx_2\ \ldots\ dx_{N-1}=0,\quad x_{N0}<x_N<x_{N1}, \tag{5.N}$$

$$\int_{x_{10}}^{x_{11}}\int_{x_{20}}^{x_{21}}\cdots\int_{x_{N-1,0}}^{x_{N-1,1}}[EF(\overline{X}_1(x_1),\ \overline{X}_2(x_2),\ \ldots,\ \overline{X}_N(x_N),\ \overline{\varphi}(\overline{x}))-$$

$$-\psi(\overline{x})]\,W_{Nk}\,dx_1dx_2\ \ldots\ dx_{N-1}\Big|_{\substack{x_N=x_{N0}\\ x_N=x_{N1}}}=0\quad(k=1,\ 2,\ \ldots,\ n_N).$$

We note that the role of the weighting functions $W_{kp}(\overline{x})$ can be taken by the unknown functions $\overline{X}_j(x_j)$ ($j = 1, 2, \ldots, k-1, k+1, \ldots, N$) (see the example). In particular, proceeding from the variational formulation, we have $W_{kp} = \partial F/\partial X_{kp}$ ($p = 1, 2, \ldots, n_k$, $k = 1, 2, \ldots, N$).

We now discuss the results. The system (5) consists of N subsystems (5.1)-(5.N) and can be used to find all the argument functions. (We call each subsystem an argument problem, since it is formulated in unknown functions of only one of the N arguments of the original problem.) However, besides the unknown functions, a given argument problem in the variable x_i also contains certain parameters obtained by the procedure of reduction of the multidimensional to a one-dimensional problem and associated with the argument functions of the other variables $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, x_{i+2}, \ldots, x_N$. This is why each of the argument problems cannot be solved separately and must be considered in interrelationship with the other argument problems. Only when analyzed simultaneously is the system of argument problems complete and usable for determining all the unknown argument functions in all the variables. Thus, besides the usual completeness within the scope of each argument problem (in the sense of correspondence between the number of equations and the number of unknowns), we also have the global completeness of the system, i.e., correspondence between the number of one-dimensional argument problems and the number of dimensions of the original multidimensional problem (N → N structure). Methods that have such a property belong to the class of methods of complete systems [7].

It must be emphasized that the governing system of the Kantorovich–Vlasov method essentially represents one of the argument problems (5) governing the functions of one argument only: x_i (N → 1 structure). It is, therefore, a special case of the general system (5) and is deduced from the latter if the argument functions in all variables of the problem except one, x_i, are eliminated at the outset in the initial stage of solution in the representation (2). It is obvious that the representation (1) so obtained is a special case of the representation (2) adopted in the method proposed here.

We now compare the projection methods discussed above in the plan of their evolution, on the one hand, and the approach proposed here, on the other. The replacement of the constant coefficients (N → 0 structure, Ritz and Galerkin procedures) by variable coefficients in one of the N arguments of the problem (N → 1 structure, Kantorovich–Vlasov method) is an important stage in the development of projection approaches. But the introduction of N families of functional coefficients in each of the variables of the problem (N → N structure, method of complete systems) is more than just the next refinement of projection methods; it is a fundamental turning point, taking the developed approach beyond the realm of projection methods. Now the method acquires wholly new attributes and possibilities [7]. The role of the individual structural components of the solution also undergoes a radical change. Thus, the center of gravity in the given method is shifted from intuitively chosen components (basic functions) to functions obtained deterministically in the course of solution of the problem [the argument functions $\{X_{ij}(x_i)\}_{j=1}^{n_i}$ ($i = 1, 2, \ldots, N$)]. Consequently, the functional foundation of projection methods, the basis $\varphi(x)$, now plays a secondary role and, as a rule, should be excluded altogether, i.e., it is required in (2) to set $\overline{\varphi}(\overline{x}) \equiv 1$. We note that projection methods lose any meaning whatsoever under such a presumption. On the other hand, the use of a basis in the given approach is justified in isolated cases where reliable information about the solution is available and it can be reliably incorporated by the introduction of basis functions.

We now give an example illustrating the effectiveness of the method developed here specifically in a situation where the basis functions are unquestionably excluded. We consider a transverse fluid flow past a long cylindrical heat-releasing element ($\rho_B \leqslant \rho \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$). The steady-state temperature distribution $T(\rho, \varphi)$ inside this element is described by the two-dimensional boundary-value problem

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial T}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 T}{\partial\theta^2}=-P(\rho,\theta),$$

$$\frac{\partial T}{\partial\rho}=0 \quad \text{for} \quad \rho=\rho_B; \quad \frac{\partial T}{\partial\rho}+\mathrm{Bi}\,T=0 \quad \text{for} \quad \rho=1; \tag{6}$$

$$\frac{\partial T}{\partial\theta}=0 \quad \text{for} \quad \theta=0,\ \pi$$

(the latter condition makes use of the symmetry of the problem with respect to θ about the cross section 0–π). Here

$$\rho=\frac{r}{r_H};\ \rho_B=\frac{r_B}{r_H};\ T=\frac{\lambda t}{q_{V\max}r_H^2};\ \mathrm{Bi}=\frac{\alpha r_H}{\lambda};\ P=\frac{q_V}{q_{V\max}}. \tag{7}$$

We solve the problem by two methods: the Kantorovich–Vlasov method of reduction to ordinary differential equations in the variable ρ and the method of complete systems developed here. The form of representation of the unknown solution in both approaches is taken as follows:

$$T_n(\rho,\theta)=\sum_{i=1}^{n}R_i(\rho)\,\Omega_i(\theta). \tag{8}$$

In the method of complete systems, however, the functions $R_i(\rho)$ and the functions $\Omega_i(\theta)$ are assumed to be unknown and must be evaluated. In the Kantorovich–Vlasov method the functions $\Omega_i(\theta)$ ($i = 1, 2, \ldots, n$) play the role of basis functions and are chosen a priori. [In the given problem the basis functions are considered to be the eigenfunctions of the original problem with respect to the variable θ: $\Omega_i(\theta) = \cos i\theta$ ($i = 1, 2, \ldots, n$).]

The governing system of argument problems for the method of complete systems, according to the preceding discussion, is written for (6) in the form

$$\sum_{i=1}^{n}\left\{\left[R_i''(\rho)+\frac{1}{\rho}R_i'(\rho)\right]J_{1ik}+\frac{1}{\rho^2}R_i(\rho)\,J_{2ik}\right\}=-J_{4k}(\rho), \tag{9.1}$$

$$R_k'(\rho_B)=0, \quad \sum_{i=1}^{n}[R_i'(1)\,J_{1ik}+R_i(1)\,J_{3ik}]=0;$$

$$\sum_{i=1}^{n}\{\Omega_i''(\theta)\,F_{1ik}-\Omega_i(\theta)\,[F_{2ik}+\mathrm{Bi}(\theta)\,R_i(1)\,R_k(1)]\}=-F_{3k}(\theta). \tag{9}$$

$$\Omega_k'(0)=\Omega_k'(\pi)=0,$$

where

$$J_{1ik}=\int_0^{\pi}\Omega_i(\theta)\,\Omega_k(\theta)\,d\theta; \quad F_{1ik}=\int_{\rho_B}^{1}\frac{1}{\rho}R_i(\rho)\,R_k(\rho)\,d\rho;$$

$$J_{2ik}=\int_0^{\pi}\Omega_i''(\theta)\,\Omega_k(\theta)\,d\theta;$$

$$J_{3ik}=\int_0^{\pi}\mathrm{Bi}(\theta)\,\Omega_i(\theta)\,\Omega_k(\theta)\,d\theta; \quad F_{2ik}=\int_{\rho_B}^{1}\rho R_i(\rho)\,R_k'(\rho)\,d\rho;$$

$$J_{4k}(\rho)=\int_0^{\pi}P(\rho,\theta)\,\Omega_k(\theta)\,d\theta; \quad F_{3k}(\theta)=\int_{\rho_B}^{1}P(\rho,\theta)\,R_k(\rho)\,d\rho.$$

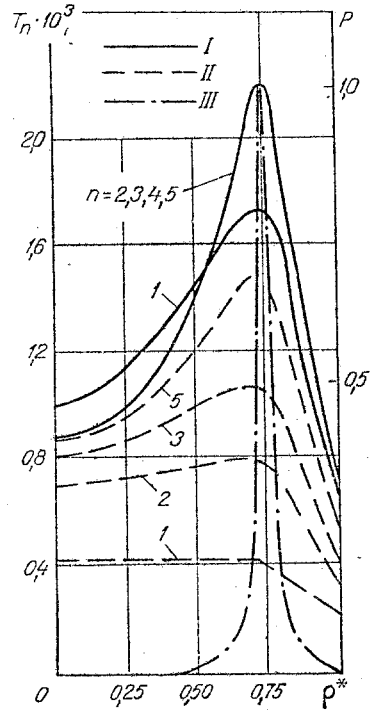

Fig. 1

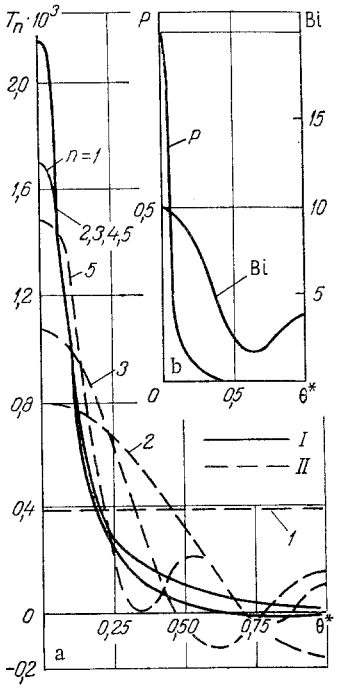

Fig. 2

Fig. 1. Distribution curves $T_n(\rho^*, \theta^*)$ for n = 1, 2, ..., 5 and $P(\rho^*, \theta^*)$ for $\theta^* = \theta/\pi = 0$. I) Proposed approach; II) Kantorovich–Vlasov method; III) $P(\rho^*, 0)$.

Fig. 2. Distribution curves $T_n(\rho^*, \theta^*)$ for n = 1, 2, ..., 5, $P(\rho^*, \theta^*)$ and $Bi(\theta^*)$ for $\rho^* = (\rho - \rho_B)/(1 - \rho_B) = 0.75$. a) T_n (0.75; θ^*); b) P(0.75; θ^*), Bi(θ^*); I) Proposed method; II) Kantorovich–Vlasov method.

The problem is solved for the following initial data: $\rho_B = 0.4$, $P(\rho, \theta) = \exp(-10(5|\rho - 0.85| + \theta))$, $Bi(\theta) = A \cos\theta + B \cos 2\theta + C \sin(\theta/2)$, $A = \alpha - B$, $B = C\sqrt{2}/2 - b$, $C = (\alpha + d + 2b)/(1 + \sqrt{2})$, $\alpha = Bi(0) = 10$, $b = Bi(\pi/2) = 2$, $d = Bi(\pi) = 4$. The governing system for the Kantorovich–Vlasov method is one of the argument problems (9), and for the method of complete systems it is the one-dimensional problem (9.1) in the variable ρ.

To compare the rates of convergence of both methods we give the results of solving the problem with the retention of various numbers n of terms in the representation (8) (Figs. 1 and 2). The solution obtained by the method of complete systems, even for n = 1, realistically reflects the qualitative pattern of the distribution of the required function $T_n(\rho, \varphi)$ in both directions, and for n = 2, we already observe practically complete convergence of the solution. For the same number of terms (n = 1, 2) the solution obtained by the Kantorovich–Vlasov method fails not only quantitatively, but even qualitatively to capture the behavior of the required temperature function, particularly in the approximating direction (Fig. 2a). Even with the retention of five (n = 5) terms in the series (8) the solution by the Kantorovich–Vlasov method has a greater error for the maximum value of the function $T_n(\rho, \varphi)$ (30%) than for one term in the method of complete systems (20%). These data, therefore, evince the high effectiveness of the proposed method in the sense of rate of convergence of the solving process.

NOTATION

x_k, independent variable; u, unknown function; D, E, differential operators; F, form of representation of the solution; φ_m, basis functions; K_j, functional coefficients; W_{kp}, weighting functions; X_{ij}, argument functions; t, temperature; q_V, output of internal heat sources; λ, thermal conductivity; α, heat-transfer coefficient; r, radial coordinate; r_B, r_H, inside and outside radii of ring (cylindrical element).

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